Realizability of Planar Point Embeddings from Angle Measurements

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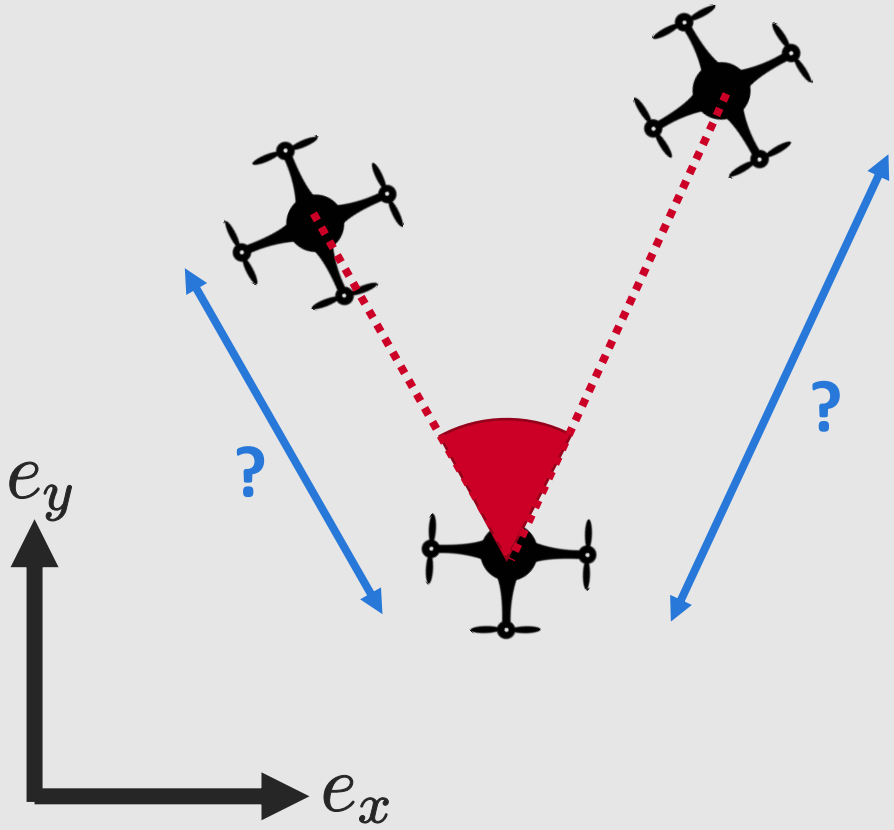
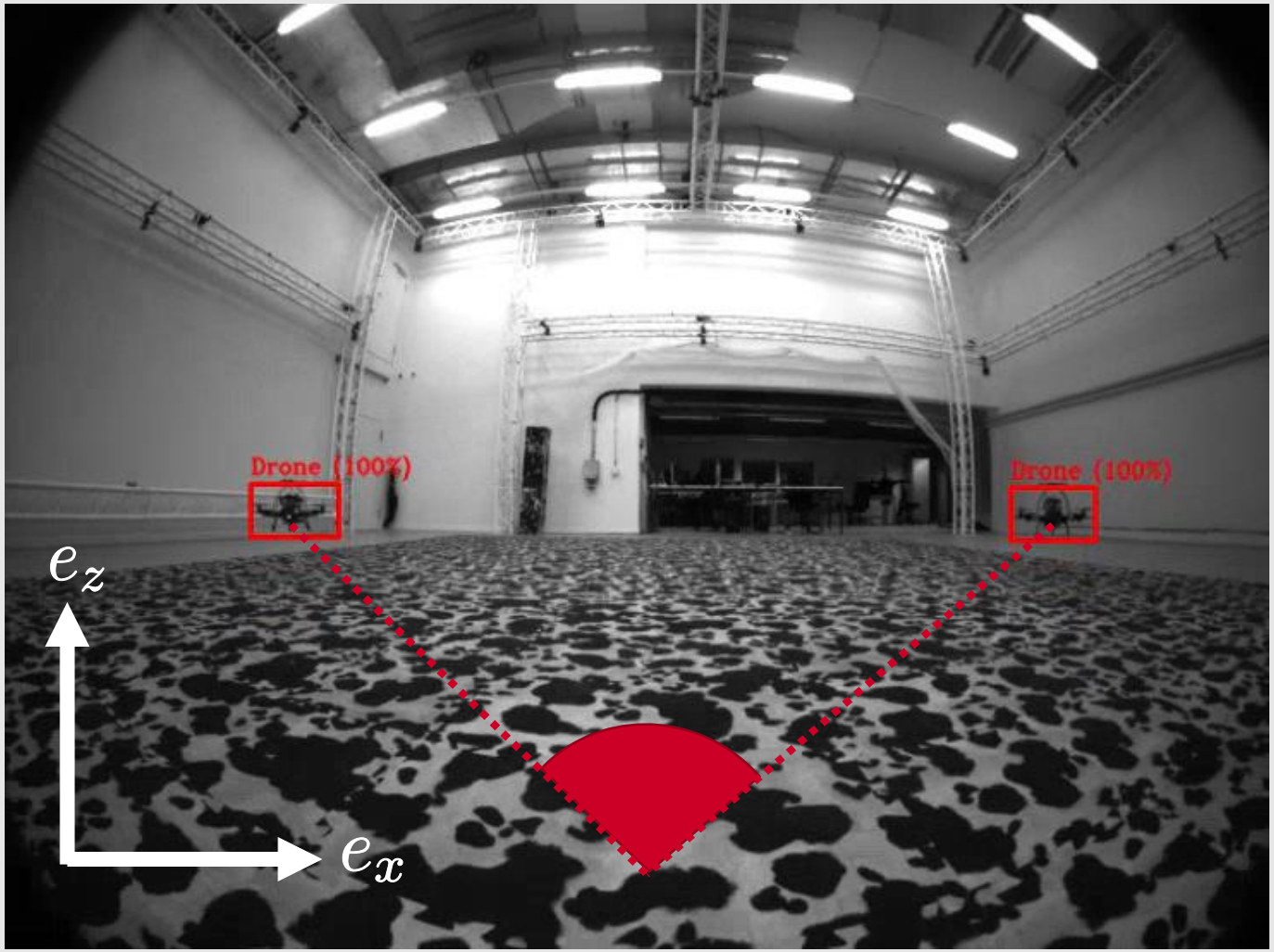
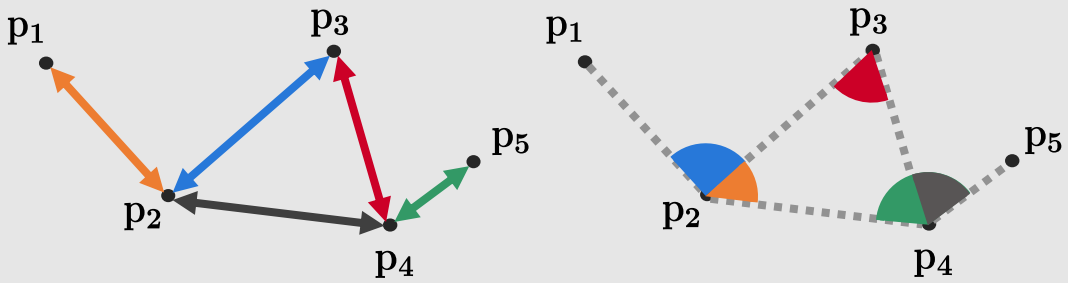




Schilling, F., Lecoeur, J., Schiano, F., & Floreano, D. (2019). Learning Vision-based Flight in Drone Swarms by Imitation. *IEEE Robotics and Automation Letters*.

Motivation | **embeddings from similarities**



Given: (dis)similarity measures between subsets of points.

Goal: find *best* relative positions of points.

Points are **physical objects**:

- Smartphones and wearables
- Robots (drones, UAVs, etc.)
- Sensors (IoT)
- Atoms in molecules
- …

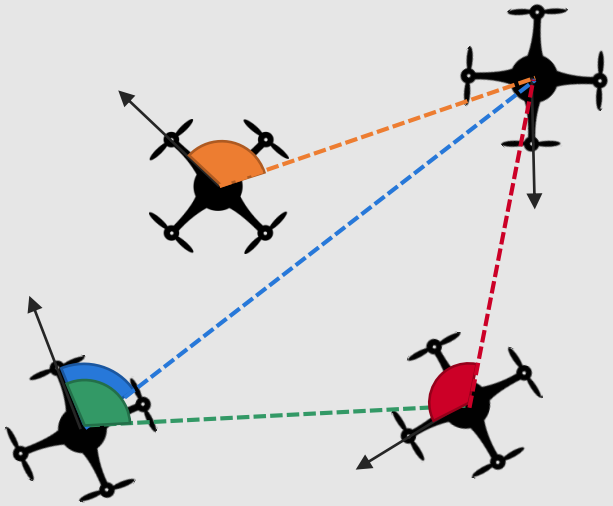
Points are **abstract:**

- Participants of experiments
- Users in social networks
- Learned low-dimensional features
- Words in a document
- …

Motivation | **applications of angles**

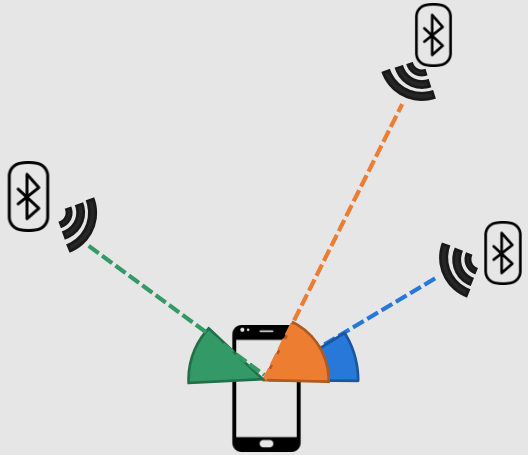
Drone swarms

monocular cameras



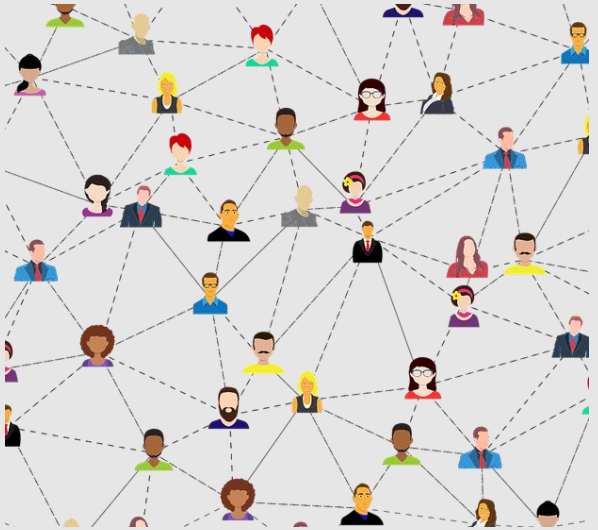
Indoor localization

AOA in Bluetooth 5.1



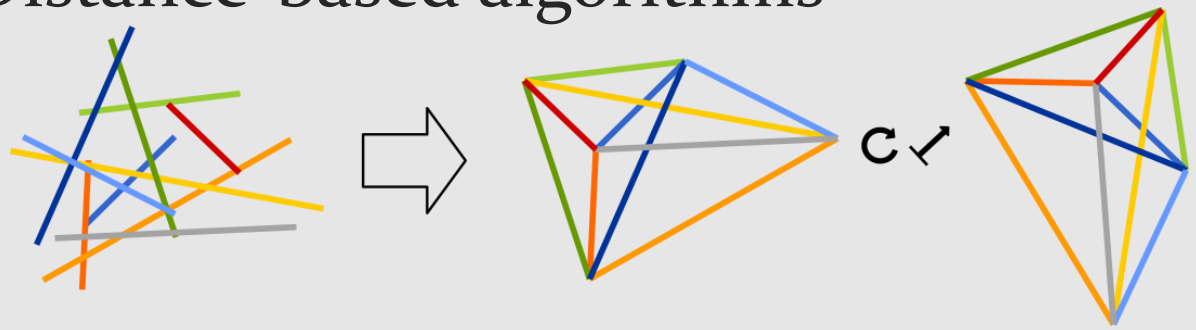
Social networks

“angle”: similarity of 2 users as seen by third user



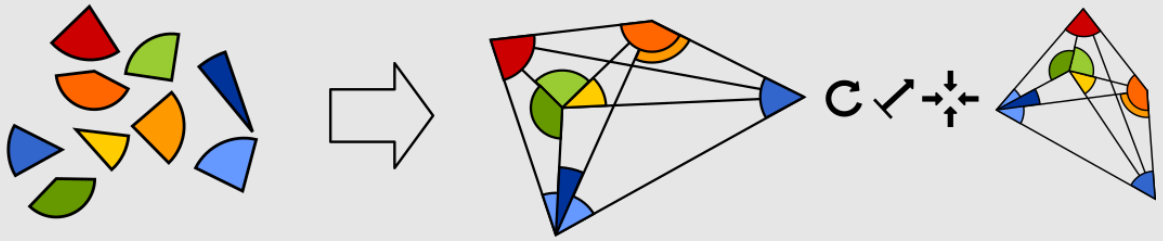
Motivation | **theory**

Distance-based algorithms



- Realizability:
  $\boldsymbol{D}$ is EDM iff $\boldsymbol{J}^\top \boldsymbol{D} \boldsymbol{J}$ is p.s.d. [1]
- Lost:
  Rigid transform ( )

Angle-based algorithms



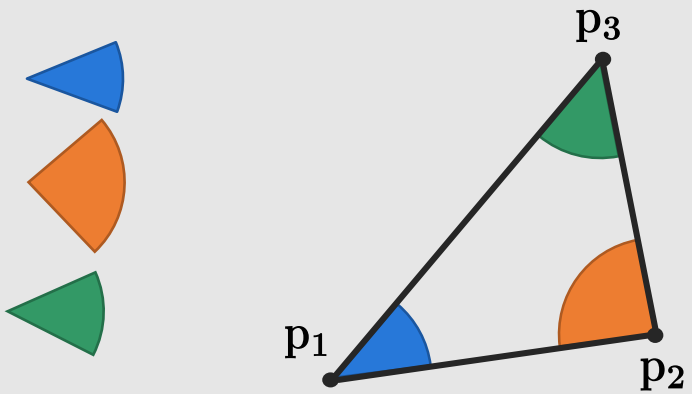
- Realizability:
  Triangular graphs: [2] general: ?
- Lost:
  Rigid transform and scale ( )

[1] Gower, J. C. (1982). Euclidean Distance Geometry. *Math. Scientist*, *7*, 1–14.

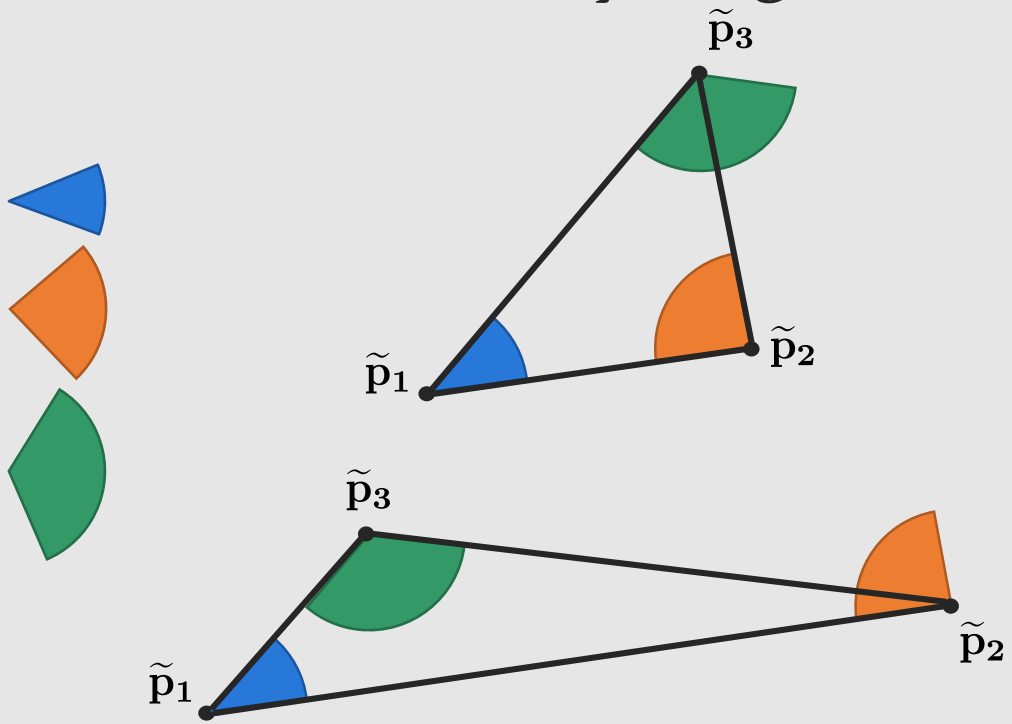
[2] Di Battista, G., & Vismara, L. (1996). Angles of Planar Triangular Graphs. *SIAM Journal on Discrete Mathematics*, *9*(3), 349–359.

Realizable angles | **simple example**

Realizable angles



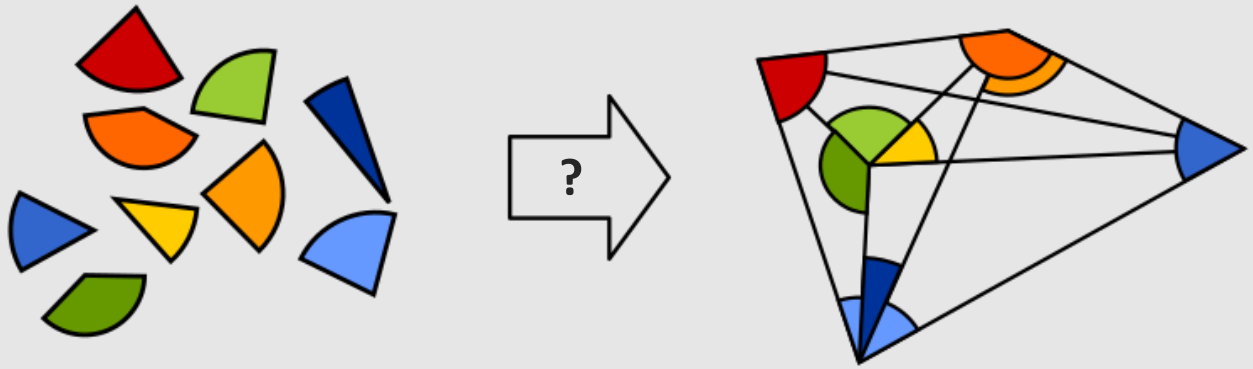
Non-realizable (noisy) angles



Realizable angles | **simple example**

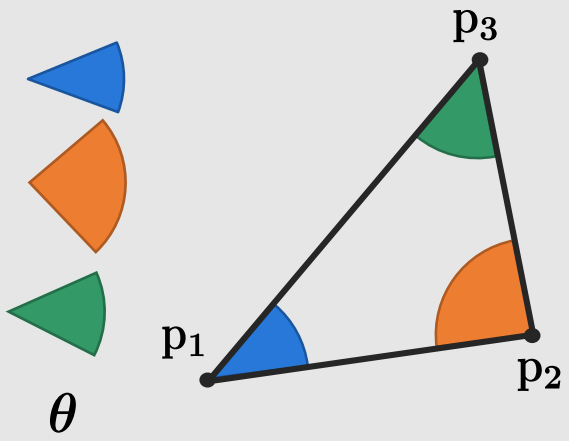
If we could characterize realizable angles, then:

- We could project the noisy angles to the **space of realizable angles.**
- Simple reconstruction algorithm would yield **unique result.**
- We could solve other problems, such as **labeling.**



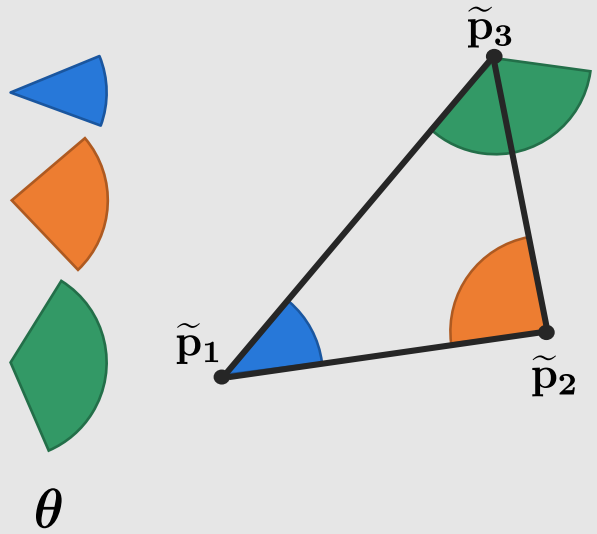
Realizable angles | **simple example revisited**

Realizable angles



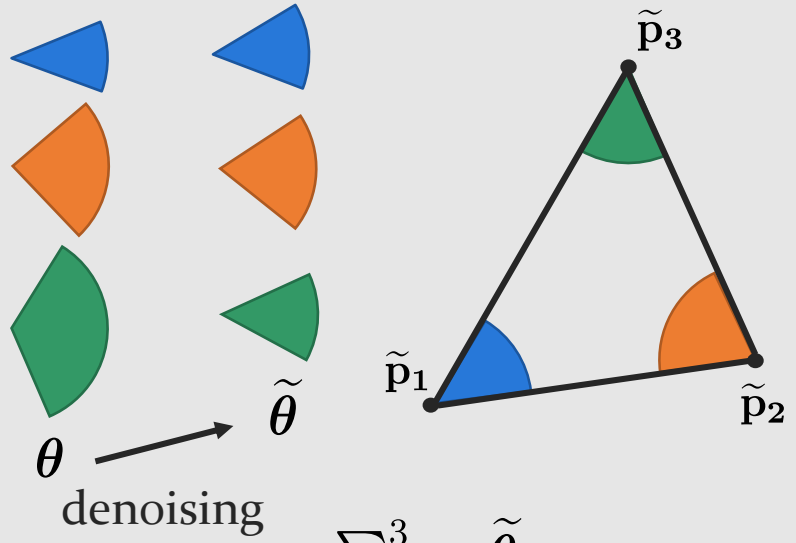
$$\sum_{m=1}^{3} \theta_m = \pi$$

Non-realizable (noisy) angles



$$\sum_{m=1}^{3} \theta_m \neq \pi$$

With realizability condition:



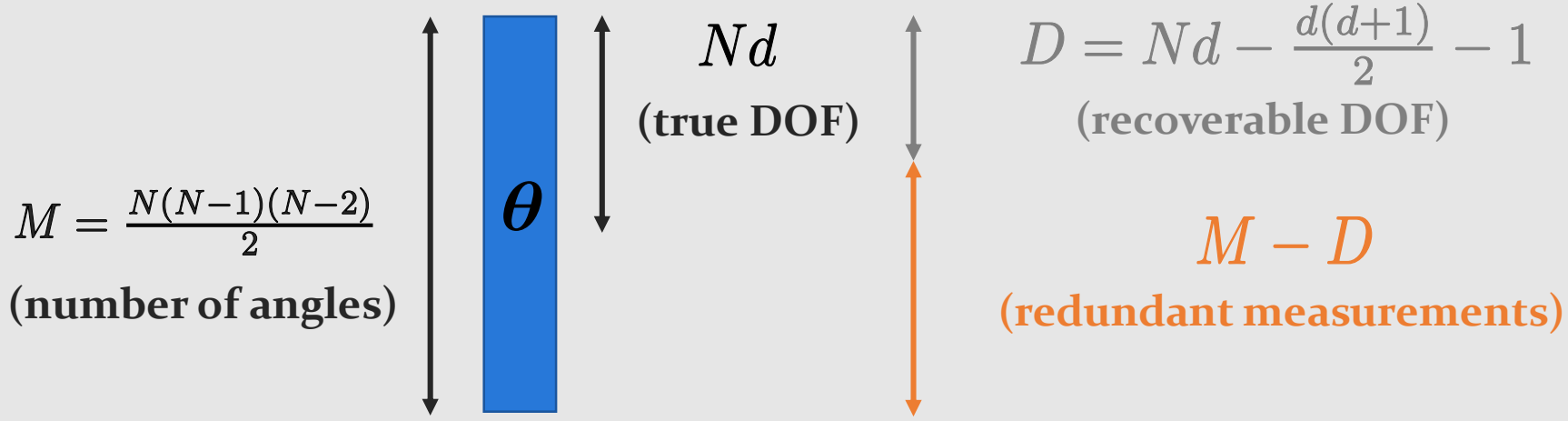
$$\sum_{m=1}^{3} \tilde{\theta}_m = \pi$$

Problem statement

- Inner angles $\theta_i(j,k) \in [0,\pi]$
- Index triplets $\mathcal{I} = \{(i,j,k)\}$, $|\mathcal{I}| = M = \frac{N(N-1)(N-2)}{2}$
- Known order $m(i,j,k) = 1,\ldots,M \quad \Rightarrow \ \boldsymbol{\theta} \in \mathbb{R}^M$
- Point embedding $\mathcal{P} = \{\boldsymbol{p_1},\ldots,\boldsymbol{p_N}\}$, $\boldsymbol{p_i} \in \mathbb{R}^d$

Definition: An angle vector $\boldsymbol{\theta}$ is realizable iff there exists a point embedding $\mathcal{P}$ such that $\angle(\boldsymbol{p_j} - \boldsymbol{p_i}, \boldsymbol{p_k} - \boldsymbol{p_i}) = \theta_{m(i,j,k)}$ for all index triplets in $\mathcal{I}$.

Realizability | **intuition**

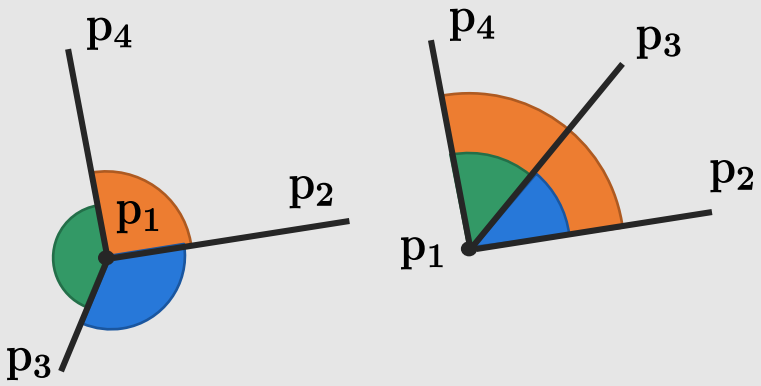


Intuition: need to constrain the redundant measurements

- Find all constraints based on trigonometric laws: **necessary conditions**
- Determine which subset of these is sufficient: **sufficient conditions**

Realizability | **necessary conditions**

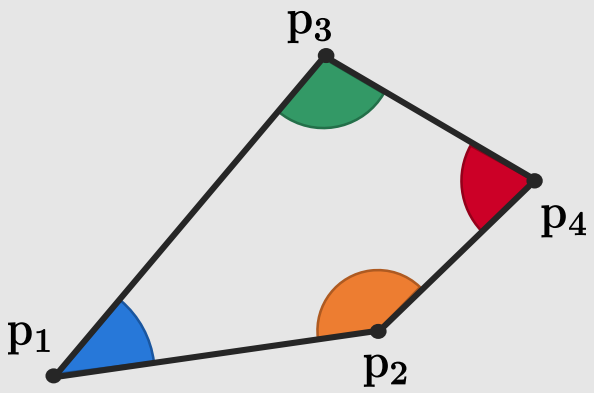
Single points



$$\sum_{k=2}^{4} \theta_1(k, k+1) = 2\pi$$

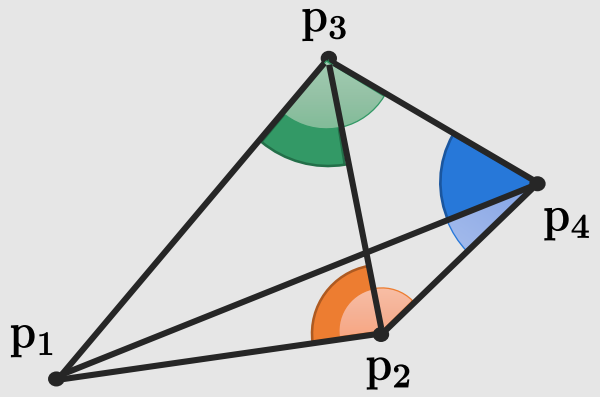
$$\sum_{k=2}^{3} \theta_1(k, k+1) = \theta_1(2, 4)$$

Convex polygons



$$\sum_{i \in P^m} \boldsymbol{\theta}_i = (m-2)\pi$$

Sine law

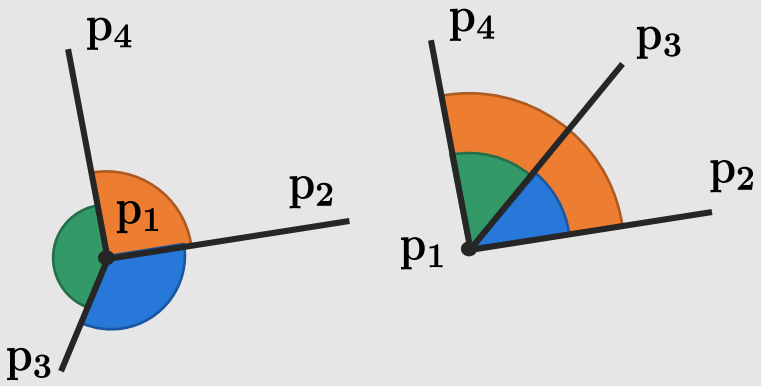


$$\frac{\sin\theta_2(1,3)}{\ell_{13}} = \frac{\sin\theta_3(1,2)}{\ell_{12}}$$

$$\frac{\sin\theta_2(1,3)}{\sin\theta_2(1,4)} \frac{\sin\theta_4(1,2)}{\sin\theta_4(1,3)} \frac{\sin\theta_3(1,4)}{\sin\theta_3(1,2)} - 1 = 0$$

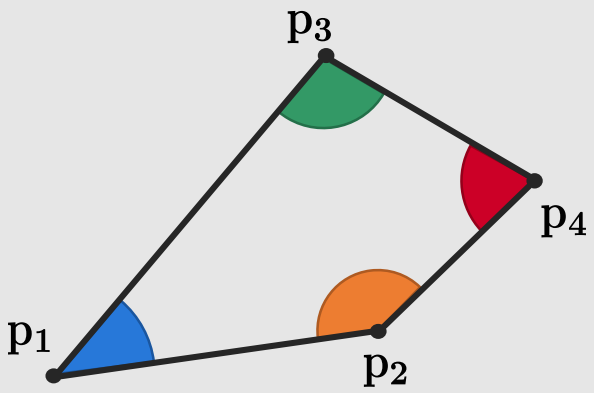
Realizability | sufficient conditions

Single points



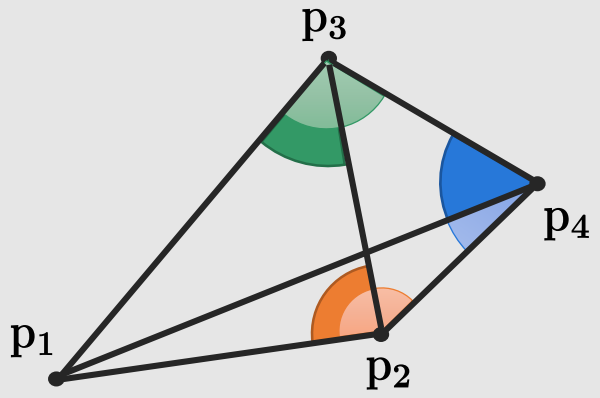
Consider all constraints.

Convex polygons



Consider all triangle constraints from one point.

Sine law



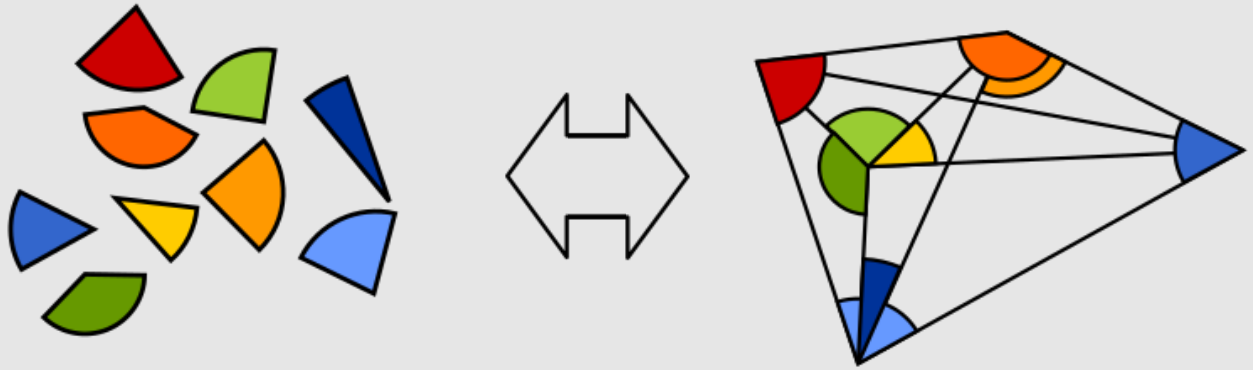
Consider one constraint per quadrilateral.

$$\boldsymbol{A\theta} = \boldsymbol{b},\ \boldsymbol{A} \in \mathbb{R}^{L \times M},\ \boldsymbol{b} \in \mathbb{R}^{L},\ L = L_{single} + L_{triangle}$$

$$f_k(\boldsymbol{\theta}) = 0,\ k = 1 \dots K$$

Realizability conditions

Conjecture: An angle vector $\boldsymbol{\theta}$ is realizable if and only if it satisfies $\boldsymbol{A\theta} = \boldsymbol{b}$ and $f_k(\boldsymbol{\theta}) = 0,\ k = 1 \dots K$.



Results | **validation of conjecture**

Conjecture: An angle vector is realizable if and only if it satisfies $\boldsymbol{A\theta = b}$ and $f_k(\boldsymbol{\theta}) = 0,\ k = 1 \dots K$.

Argument 1 : constraints add up and are independent!

$$L_{single} + L_{linear} + K = N \sum_{i=1}^{N-2} i + \binom{N-1}{2} + \binom{N}{4} = M - D$$

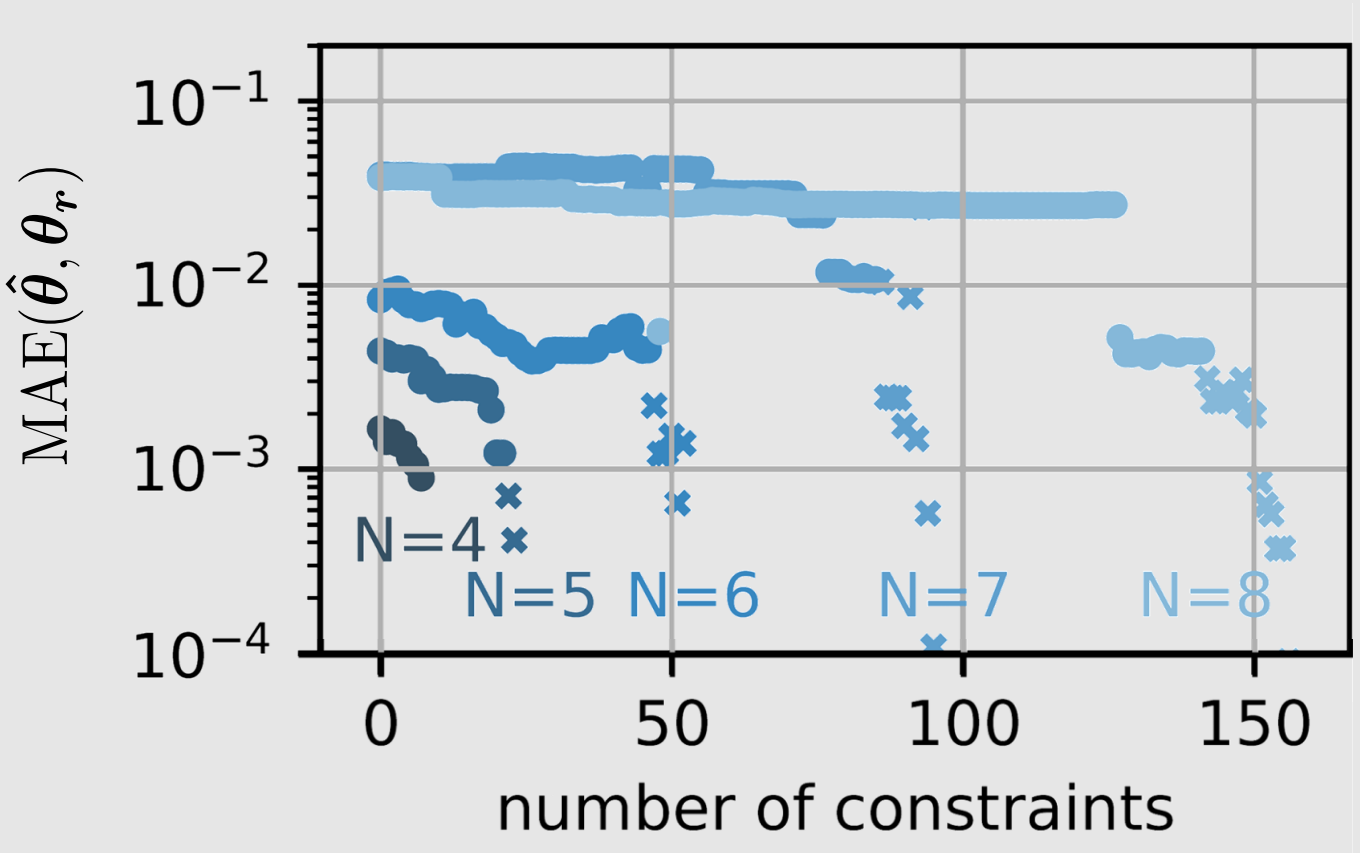
Results | **validation of conjecture**

Conjecture: An angle vector is realizable if and only if it satisfies $\boldsymbol{A\theta} = \boldsymbol{b}$ and $f_k(\boldsymbol{\theta}) = 0,\ k = 1 \dots K$.

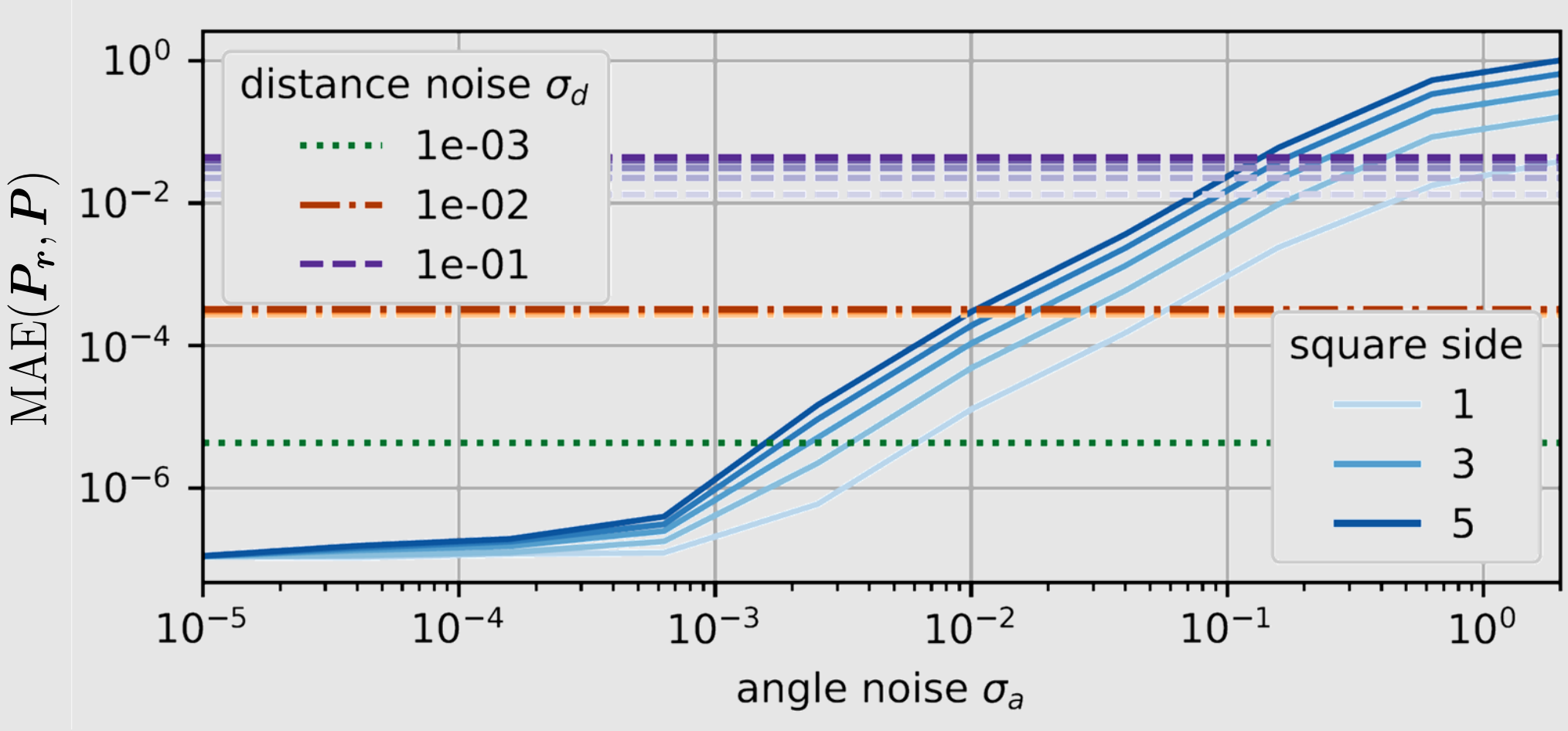
Argument 2 : can find point set from angles satisfying Conjecture!

1. Get realizable angles:
$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \mathbb{R}^M}{\arg\min} ||\widetilde{\boldsymbol{\theta}} - \boldsymbol{\theta}||,$$
$$\text{such that } \boldsymbol{A\theta} = \boldsymbol{b},\ f_k(\boldsymbol{\theta}) = 0 \text{ for } k = 1 \dots K.$$
2. Create embedding $\mathcal{P}_r$ using simple build-up algorithm $\rightarrow \boldsymbol{\theta_r}$
3. Discrepancy error, should be zero: $\text{MAE}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta_r})$

Results | **validation of conjecture**



Results | **reconstruction performance**



Conclusion and future work

Summary

- Characterized necessary conditions for realizability.
- Conjectured and successfully tested sufficient conditions.
- Proposed angle-based denoising and point recovery algorithm.

Open questions

- Prove sufficiency of conditions.
- Design branch+prune method for solving the labeling problem.

Thank you!

Code available at: github.com/duembgen/AngleRealizability
Paper available at: infoscience.epfl.ch/record/274650

Happy to answer any questions: frederike.duembgen@epfl.ch